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APPLICATION OF PONTRYAGIN'S MAXIMUM PRINCIPLE FOR MINIMUM WEIGHT DESIGN OF RIGID–PLASTIC CIRCULAR PLATES

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Abstract—The problem of minimum weight design of sandwich-type circular plates for the complementary condition that the thicknesses of the carrying layers can never be smaller than a specified value h_0, is considered. The material of the plate is regarded as rigid–plastic; for the plastic yield condition any piecewise-linear yield condition can be taken.

The problem in question is solved with the aid of the Pontryagin's maximum principle. Possible optimal régimes are found. The mechanical meaning of the adjoint system is cleared up. The case where the material of the plate has different yield stresses in tension and compression is discussed in detail. The minimum weight design problem for simply supported plates under uniformly distributed pressure is solved.

INTRODUCTION

THE minimum weight problem of rigid–plastic plates with sandwich cross sections has been discussed in many papers. Usually the outer (carrying) layers are considered variable, the weight of the central layer (core) is neglected, the whole thickness of the plate is constant. Under these assumptions by using the Tresca's yield condition the minimum weight problem of circular and annular plates was solved in [1–3]. The same problem for the Mises yield condition was studied in [4–5]. In the majority of these solutions the thicknesses of the carrying layers at certain cross sections were found to be zero. This is an impractical result since such sections cannot transmit shear forces. With the purpose to get solutions which are free from this shortcoming in the present paper the plates with the complementary condition, that the thicknesses of the carrying layers can never be smaller than a specified value h_0 are considered. The material of the plate is regarded rigid–plastic (without strain hardening); for the yield condition any piece-wise linear yield condition can be taken (thus our results are also valid for anisotropic plates).

The problem is solved with the aid of the Pontryagin's maximum principle. Possible optimal regimes are found (Section 2). They can be divided into three following groups; 1. both carrying layers have minimal thicknesses h_0, 2. one layer has minimal thickness h_0, the thickness of the other layer is variable, and 3. both layers have variable thicknesses.

A more detailed analysis is given for a material having different yield stresses in tension and compression (the Prager's model is used). Synthesis of the optimal solutions, which were found in Section 2, is carried out for a simply supported plate under uniform pressure (Section 4–5).

1. BASIC EQUATIONS

Let us consider a simply supported plate of radius R under the lateral load of magnitude q. The equilibrium equations of the plate are as follows

$$\frac{\mathrm{d}T_1}{\mathrm{d}\rho} = \frac{T_2 - T_1}{\rho}, \qquad \frac{\mathrm{d}M_1}{\mathrm{d}\rho} = \frac{M_2 - M_1}{\rho} - \tfrac{1}{2}qR^2\rho, \tag{1.1}$$

where $\rho = r/R$; T_1, T_2, M_1, M_2 stand for membrane forces and bending moments in the radial and circumferential directions.

The plate has a sandwich cross section consisting of three layers (Fig. 1). Let us mark all the quantities for the upper and lower layer of the cross section with superscripts "$-$" and "$+$", respectively. The thicknesses of the layers h^- and h^+ are regarded different and variable with co-ordinate ρ. Besides, these quantities are limited from below, that is $h^-(\rho) \geq h_0$, $h^+(\rho) \geq h_0$. The whole thickness of the plate D is regarded invariable.

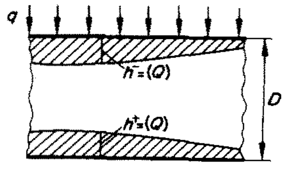

FIG. 1. Sandwich-type cross section.

If $\sigma_1^-, \sigma_2^-, \sigma_1^+, \sigma_2^+$ are the stresses in the carrying layers, one can calculate the quantities T_1, T_2, M_1, M_2 from the formulae

$$\begin{aligned} T_i &= h_0(\sigma_i^+ u^+ + \sigma_i^- u^-) \\ M_i &= \tfrac{1}{2}Dh_0(\sigma_i^+ u^+ - \sigma_i^- u^-), \qquad (i = 1, 2) \end{aligned} \tag{1.2}$$

where $u^- = h^-/h_0 \geq 1$, $u^+ = h^+/h_0 \geq 1$.

Since we are going to design a plate the outer layers of which would have a minimum weight, the performance index will be

$$J = \int_0^1 (u^- + u^+)\rho \,\mathrm{d}\rho = \min. \tag{1.3}$$

We shall take quantities T_1, M_1 and ρ for state constraints. The Pontryagin's Hamiltonian has the form

$$H = \psi_0(u^- + u^+)\rho + \frac{\psi_1}{\rho}(T_2 - T_1) + \frac{\psi_2}{\rho}(M_2 - M_1 - \tfrac{1}{2}qR^2\rho^2) + \psi_3. \tag{1.4}$$

Here ψ_0, ψ_1, ψ_2, ψ_3 are the Lagrangian multipliers; ψ_0 is a nonpositive constant, ψ_1, ψ_2, ψ_3 can be found from the adjoint system

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\rho} = -\frac{\partial H}{\partial T_1}, \qquad \frac{\mathrm{d}\psi_2}{\mathrm{d}\rho} = -\frac{\partial H}{\partial M_1}, \qquad \frac{\mathrm{d}\psi_3}{\mathrm{d}\rho} = -\frac{\partial H}{\partial \rho}. \tag{1.5}$$

According to the maximum principle such admissible control constraints should be found which give the Hamiltonian a maximum value for $\forall\rho \in [0, 1]$.

The following boundary conditions belong to equations (1.1) and (1.5). At the center $\rho = 0$ one has $\psi_1 = \psi_2 = 0$. At the boundary $\rho = 1$ we can have two variants of boundary conditions: (a) $M_1 = 0$ and $T_1 = 0$ or (b) $M_1 = 0, v = 0, \psi_2 = 0$ (here v is the rate of radial displacement).

2. OPTIMAL REGIMES

Let us suppose that the yield condition of the material has the form as shown in Fig. 2. It can be easily demonstrated that the stresses in the carrying layers $\sigma_1^\pm$, $\sigma_2^\pm$ for optimal régimes must correspond to the sides or to the vertices of the yield polygon in Fig. 2. To prove this statement let us regard quantities T_2, M_2, $u^\pm$ for control constraints. It follows from (1.5) that $\psi_1 = C_1\rho$, $\psi_2 = C_2\rho$. The maximum of the Hamiltonian is realized when T_2 and M_2 have extremal values; it follows from the formulae (1.2) that these extremal values can be obtained only on the boundary of the yield curve in Fig. 2, which was to be proved.

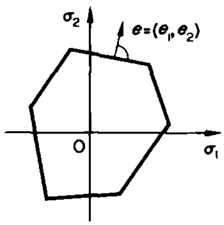

FIG. 2. Yield polygon.

Now we shall examine some special cases.

First case

Let us assume that the stress states $\sigma_1^\pm$, $\sigma_2^\pm$ correspond to two sides of the yield polygon in Fig. 2 and the equations of these sides are

$$\sigma_1^- = a\sigma_2^- + b, \qquad \sigma_1^+ = e\sigma_2^+ + f. \tag{2.1}$$

Here a, b, e, f are constants. It is assumed that $a \neq e$, $a \neq 0$, $e \neq 0$.

Making use of the equations (1.2) and (2.1) one obtains

$$\begin{aligned} T_2 &= -h_0\left(\frac{b}{a}u^- + \frac{f}{e}u^+\right) + \frac{1}{2a}\left(T_1 - \frac{2}{D}M_1\right) + \frac{1}{2e}\left(T_1 + \frac{2}{D}M_1\right) \\ \frac{2}{D}M_2 &= h_0\left(\frac{b}{a}u^- - \frac{f}{e}u^+\right) - \frac{1}{2a}\left(T_1 - \frac{2}{D}M_1\right) + \frac{1}{2e}\left(T_1 + \frac{2}{D}M\right) . \end{aligned} \tag{2.2}$$

As the quantities T_2, M_2 can now be calculated from the formulae (2.2), the independent control constants are only u^- and u^+. Using the equations (1.4) and (2.2) the two first formulae of the system (1.5) obtain the form

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\rho} = \frac{A}{\rho}\psi_1 + B\frac{D}{2\rho}\psi_2, \qquad \frac{D}{2}\frac{\mathrm{d}\psi_2}{\mathrm{d}\rho} = \frac{B}{\rho} + \frac{AD}{2\rho}\psi_2, \tag{2.3}$$

where

$$A = 1 - \frac{1}{2a} - \frac{1}{2e}, \qquad B = \frac{1}{2a} - \frac{1}{2e}.$$

The integrals of these equations are

$$\psi_1 = C_1\rho^{A+B} + C_2\rho^{A-B}, \qquad \frac{D}{2}\psi_2 = C_1\rho^{A+B} - C_2\rho^{A-B}. \tag{2.4}$$

Since $u^{\pm} \geq 1$, the maximum of the Hamiltonian (1.4) is guaranteed only when $\partial H/\partial u^{\pm} \leq 0$, This requirement leads us to the following inequalities

$$\psi_0 \leq \frac{2C_2 h_0 b}{a}\rho^{A-B-2}, \qquad \psi_0 \leq \frac{2C_1 h_0 f}{e}\rho^{A+B-2}. \tag{2.5}$$

These results remain valid when one (or both) of the straight lines (2.1) are parallel to the σ_1-axis; then we have $a \to \infty$ or $e \to \infty$ respectively, but the relations b/a and f/e remain finite. As to the cases $a = 0$ or $e = 0$ (one of the sides is parallel to the σ_2-axis), they demand a special analysis.

In all subcases which were examined above, the maximum of the Hamiltonian (1.4) took place only in case of $u^{\pm} \equiv 1$; consequently both carrying layers must have the minimum thickness h_0.

Second case

Here the stress states $\sigma_1^{\pm}$, $\sigma_2^{\pm}$ correspond to a side and a vertex of the yield polygon in Fig. 2. Let the co-ordinates of the vertex be $(\sigma_1^+,\ \sigma_2^+)$ and the equation of the side $\sigma_1^- = a\sigma_2^- + b$. Now we get the following formulae for T_2 and M_2:

$$\begin{aligned} T_2 &= \frac{1}{2}\left(\frac{1}{a} + \frac{\sigma_2^+}{\sigma_1^+}\right)T_1 + \frac{1}{D}\left(\frac{\sigma_2^+}{\sigma_1^+} - \frac{1}{a}\right)M_1 - \frac{b}{a}h_0 u^- \\ \frac{2}{D}M_2 &= \frac{1}{2}\left(\frac{\sigma_2^+}{\sigma_1^+} - \frac{1}{a}\right)T_1 + \frac{1}{D}\left(\frac{\sigma_2^+}{\sigma_1^+} + \frac{1}{a}\right)M_1 + \frac{b}{a}h_0 u^-. \end{aligned} \tag{2.6}$$

The dimensionless thickness u^+ can be calculated from the formula

$$u^+ = \frac{1}{2\sigma_1 + h_0}\left(T_1 + \frac{2}{D}M_1\right). \tag{2.7}$$

In view of the equations (2.6) and (2.7) the adjoint system (1.5) takes the form

$$\begin{aligned} \frac{\mathrm{d}\psi_1}{\mathrm{d}\rho} &= -\frac{\psi_0\rho}{2h_0\sigma_1^+} + A_1\frac{\psi_1}{\rho} + B_1 D\frac{\psi_2}{2\rho} \\ \frac{D}{2}\frac{\mathrm{d}\psi_2}{\mathrm{d}\rho} &= -\frac{\psi_0\rho}{2h_0\sigma_1^+} + B_1\frac{\psi_1}{\rho} + A_1 D\frac{\psi_2}{2\rho}, \end{aligned} \tag{2.8}$$

where

$$A_1 = 1-\frac{1}{2}\left(\frac{\sigma_2^+}{\sigma_1^+}+\frac{1}{a}\right), \qquad B = \frac{1}{2}\left(\frac{1}{a}-\frac{\sigma_2^+}{\sigma_1^+}\right). \tag{2.9}$$

The solution of the equations (2.8) gives

$$\begin{aligned} \psi_1 &= -\frac{\psi_0\rho^2}{4h_0\sigma_1^+}+C_1\rho^{A_1+B_1}+C_2\rho^{A_1-B_1} \\ \frac{D}{2}\psi_2 &= -\frac{\psi_0\rho^2}{4h_0\sigma_1^+}+C_1\rho^{A_1+B_1}-C_2\rho^{A_1-B_1}. \end{aligned} \tag{2.10}$$

The subcase $\sigma_1^+ = 0$ needs a complementary analysis.

Since in the second case quantities T_2, M_2 and u^+ can be found from equations (2.6) and (2.7), only u^- can be taken for the control constraint. Since $u^- \geq 1$, according to the maximum principle it must be

$$\frac{\partial H}{\partial u_1} = \psi_0\rho+\frac{b}{a}h_0\left(-\frac{\psi_1}{\rho}+\frac{D}{2}\frac{\psi_2}{\rho}\right) \leq 0.$$

In view of (2.10) this condition gives

$$\psi_0 \leq \frac{2C_2h_0b}{a}\rho^{A_1-B_1-2}. \tag{2.11}$$

Consequently in the second case we have $u^- \equiv 1$ or $h^- \equiv h_0$. The thickness of the other layer is calculated from (2.7).

Third case

In this case the stress states in both carrying layers correspond to the vertices of the yield polygon. If the co-ordinates of the vertices are (σ_1^-, σ_2^-), (σ_1^+, σ_2^+), we get the following equations according to (1.2):

$$\begin{aligned} T_2 &= \frac{\sigma_2^+}{2\sigma_1^+}\left(T_1+\frac{2}{D}M_1\right)+\frac{\sigma_2^-}{2\sigma_1^-}\left(T_1-\frac{2}{D}M_1\right) \\ \frac{2}{D}M_2 &= \frac{\sigma_2^+}{2\sigma_1^+}\left(T_1+\frac{2}{D}M_1\right)-\frac{\sigma_2^-}{2\sigma_1^-}\left(T_1-\frac{2}{D}M_1\right), \end{aligned} \tag{2.12}$$

$$\begin{aligned} u_1 &= \frac{1}{2h_0\sigma_1^-}\left(T_1-\frac{2}{D}M_1\right) \\ u_2 &= \frac{1}{2h_0\sigma_1^+}\left(T_1+\frac{2}{D}M_1\right). \end{aligned} \tag{2.13}$$

Since all the quantities T_2, M_2, u_1, u_2 are calculated from (2.12) and (2.13) in this case we have no control constraints. The adjoint system (1.5) is

$$\begin{aligned} \frac{d\psi_1}{d\rho} &= -\frac{\psi_0\rho}{2h_0}\left(\frac{1}{\sigma_1^+}+\frac{1}{\sigma_1^-}\right)+A_2\frac{\psi_1}{\rho}+B_2D\frac{\psi_2}{2\rho} \\ \frac{D}{2}\frac{d\psi_2}{d\rho} &= -\frac{\psi_0\rho}{2h_0}\left(\frac{1}{\sigma_1^+}-\frac{1}{\sigma_1^-}\right)+B_2\frac{\psi_1}{\rho}+A_2D\frac{\psi_2}{2\rho}, \end{aligned} \tag{2.14}$$

where

$$A_2 = 1 - \frac{\sigma_2^+}{2\sigma_1^+} - \frac{\sigma_2^-}{2\sigma_1^-}, \qquad B_2 = \frac{1}{2}\left(\frac{\sigma_2^-}{\sigma_1^-} - \frac{\sigma_2^+}{\sigma_1^+}\right). \tag{2.15}$$

The integrals of the system (2.14) have an analogical form to (2.10).

As the thicknesses of both carrying layers are variable, they can be calculated from (2.13).

Thus we have found all optimal régimes for the yield polygon in Fig. 2. These régimes must be synthesized for the minimum weight design of a circular plate. Besides, the following conditions should be fulfilled: 1. the boundary conditions for T_1, M_1, ψ_1, ψ_2 at $\rho = 0$ and $\rho = 1$, 2. the continuity conditions for T_1, M_1, ψ_1, ψ_2 at the contact points of different régimes, 3. the inequalities (2.5) and (2.11). As to the third Lagrange multiplier ψ_3, in the given problem we have not any boundary conditions for it and the last equation of the system (1.5) can always be integrated so that ψ_3 is continuous for $\rho \in [0, 1]$. Consequently this quantity is of no importance in solving the present problem.

3. THE MECHANICAL MEANING OF THE LAGRANGE MULTIPLIERS

We shall proceed from the associated flow law, according to which the plastic flow vector is perpendicular to the sides of the polygon in Fig. 2. Let $e_1^\pm$ and $e_2^\pm$ be the strain rates of the carrying layers; these quantities are related to displacement rates v and w by the formulae

$$e_1^\pm = \frac{1}{R}\frac{\mathrm{d}v}{\mathrm{d}\rho} \mp \frac{D}{2R^2}\frac{\mathrm{d}^2w}{\mathrm{d}\rho^2}, \qquad e_2^\pm = \frac{1}{R}\frac{v}{\rho} \mp \frac{D}{2R^2\rho}\frac{\mathrm{d}w}{\mathrm{d}\rho}. \tag{3.1}$$

As to the first case of Section 2, it follows from the associated flow law and from the equations (2.1) that

$$ae_1^- + e_2^- = 0, \qquad ee_1^+ + e_2^+ = 0. \tag{3.2}$$

Making use of (3.1) the equations (3.2), after some algebraic transformations, can be presented in the form

$$\frac{\mathrm{d}}{\mathrm{d}\rho}(v\rho) = Av - B\frac{D}{2}\frac{\mathrm{d}w}{\mathrm{d}\rho}, \qquad \frac{D}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}(w\rho) = -Bv + A\frac{D}{2}\frac{\mathrm{d}v}{\mathrm{d}\rho}, \tag{3.3}$$

where

$$A = 1 - \frac{1}{2a} - \frac{1}{2e}, \qquad B = \frac{1}{2a} - \frac{1}{2e}.$$

These results completely coincide with (2.3) if we take

$$\psi_1 = \pm\rho v, \qquad \psi_2 = \mp\rho\frac{\mathrm{d}w}{\mathrm{d}\rho}. \tag{3.4}$$

From the double signs in (3.4) one has to choose such signs, that the strain vector would be directed outward of the yield polygon.

Now let us examine the second case of Section 2. For the side $\sigma_1^- = a\sigma_2^- + b$ the flow rule gives $ae_1^- + e_2^- = 0$. It is known from the theory of minimum weight design (see e.g. [2]) that on the layer of variable thickness the rate of plastic dissipation Δ is constant, consequently

$$\Delta = \sigma_1^+ e_1^+ + \sigma_2^+ e_2^+ = K > 0. \tag{3.5}$$

Putting the values of the strain rates $e_1^\pm$, $e_2^\pm$ from (3.1) into (3.5) and into the equation $ae_1^- + e_2^- = 0$, one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho v) &= \frac{KR}{2\sigma_1^+}\rho + A_1 v - B_1 \frac{D}{2}\frac{\mathrm{d}w}{\mathrm{d}\rho} \\ \frac{D}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}w}{\mathrm{d}\rho}\right) &= -\frac{KR}{2\sigma_1^+}\rho - B_1 v + A_1 \frac{D}{2}\frac{\mathrm{d}w}{\mathrm{d}\rho}. \end{aligned} \tag{3.6}$$

Quantities A_1, B_1 are defined by the formulae (2.9).

The comparison of expressions (3.6) and (2.8) leads us again to (3.4); besides, we have $\psi_0 = -KRh_0 < 0$. An analogical analysis, which was carried out for the third case, showed that the formulas are valid here as well. So it can be concluded that formulae (3.4) hold good for every optimal régime.

4. OPTIMAL DESIGN OF PLATES WITH BOUNDARY CONDITIONS $T_1(1) = 0$

In this part of the paper it is assumed that the yield polygon has the form shown in Fig. 3. Such a hexagon was proposed by Prager for materials which have different yield stresses in tension and compression $\sigma_s^\pm$. In the following analysis we shall examine only the case $\gamma = \sigma_s^-/\sigma_s^+ \geq 1$ (for the sake of conciseness).

Let us assume that the following plastic régimes in Fig. 3 are realized for the plate of minimum weight* 1. $D-A$ for $\rho \in [\rho_1, \rho_1)$, 2. $D-BA$ for $\rho \in (\rho_1, \rho_2)$, 3. $DE-BA$ for $(\rho_2, 1]$.

The formulae of Section 2 will be valid provided we carry out the following substitutions $b/a = \sigma_s^-$, $f/e = -\sigma_s^+$, $a \to \infty$, $e \to \infty$, $\sigma_1^+ = \sigma_2^+ = \sigma_s^+$, $\sigma_1^- = \sigma_2^- = -\sigma_s^-$. Putting these values into (2.2), (2.6) and (2.12), we obtain

$$T_2 = T_1, \qquad M_2 = M_1, \qquad \text{for} \qquad \rho \in [0_1 \rho_1),$$

$$T_2 = \tfrac{1}{2}T_1 - \frac{1}{D}M_1 - \sigma_s^- h_0, \qquad M_2 = \frac{D}{4}T_1 + \tfrac{1}{2}M_1 + \sigma_s^- h_0, \quad \text{for} \quad \rho \in (\rho_1, \rho_2)$$

$$T_2 = h_0(\sigma_s^+ - \sigma_s^-), \qquad M_2 = \frac{D}{2}h_0(\sigma_s^- + \sigma_s^+), \quad \text{for} \quad \rho \in (\rho_2, 1].$$

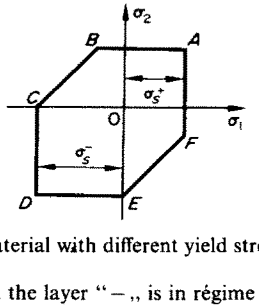

FIG. 3. The Prager's model for material with different yield stresses in tension and compression.

* The designation $DE-BA$ shows that the layer "– „ is in régime DE and the layer "+ „ in BA.

Using these expressions let us integrate the equilibrium equations (1.1). The integration constants can be found from the boundary conditions $T_1(1) = M_1(1) = 0$ and from the continuity conditions for T_1 and M_1 at $\rho = \rho_1$, $\rho = \rho_2$. Fulfilling still the conditions $u^-(\rho_1) = 1, u^+(\rho_2) = 1$ we get the equations

$$Q(1-\rho_1^3) = 1, \qquad Q\gamma(1-\rho_2^3) = 1, \tag{4.1}$$

where

$$Q = \frac{qR^2}{6Dh_0\sigma_s^-}.$$

For a given nondimensional load Q the quantities ρ_1, ρ_2 can be found from equations (4.1).

Since $A = 1, B = 0, A_1 = \frac{1}{2}, B_1 = -\frac{1}{2}, A_2 = B_2 = 0$ the formulae (2.4), (2.10), (2.14) give

$$\psi_1 = C_1 - \frac{\psi_0\rho^2}{4h_0}\left(\frac{1}{\sigma_s^+} - \frac{1}{\sigma_s^-}\right), \quad \frac{D}{2}\psi_2 = C_2 - \frac{\psi_0\rho^2}{4h_0}\left(\frac{1}{\sigma_s^+} + \frac{1}{\sigma_s^-}\right), \quad \text{for } \rho\in[0_1\rho_1),$$

$$\psi_1 = -\frac{\psi_0\rho^2}{4h_0\sigma_s^+} + C_3 + C_4\rho, \quad \frac{D}{2}\psi_2 = -\frac{\psi_0\rho^2}{4h_0\sigma_s^+} + C_3 - C_4\rho \quad \text{for } \rho\in(\rho_1, \rho_2),$$

$$\psi_1 = C_5\rho, \quad \frac{D}{2}\psi_2 = C_6\rho, \quad \text{for } \rho\in(\rho_2, 1].$$

Constants $C_1 - C_6$ can be determined from the boundary conditions $\psi_1(0) = \psi_2(0) = 0$ and from the continuity conditions for ψ_1, ψ_2 at $\rho = \rho_1$ and $\rho = \rho_2$. As to inequalities (2.5) and (2.11), it is easy to verify that they are always satisfied. Thus our solution fulfills all the conditions which were set up in Sections 1–2 and actually presents a plate of minimum weight. As to the nondimensional thicknesses $u^\pm$, then in view of (2.7) and (2.13) they can be calculated from the formulae

$$\begin{aligned} u^- &= 1+\tfrac{3}{2}Q(\rho_1^2-\rho^2), \quad \text{for } \rho\in[0_1\rho_1] \quad \text{and} \quad u^- \equiv 1, \quad \text{for } \rho\in[\rho_1, 1] \\ u^+ &= 1+\tfrac{3}{2}Q\gamma(\rho_2^2-\rho^2), \quad \text{for } \rho\in[0_1\rho_2] \quad \text{and} \quad u^+ \equiv 1, \quad \text{for } \rho\in[\rho_2, 1]. \end{aligned} \tag{4.2}$$

5. PLATE DESIGN FOR BOUNDARY CONDITION $v(1) = 0$

For this problem the following plastic régimes are realized: 1. $D-A$ for $\rho\in[0_1\rho_1)$, 2. $DE-A$ for $\rho\in(\rho_1, \rho_2)$, 3. $DE-BA$ for $\rho\in(\rho_2, 1]$. The boundary and continuity conditions are the same as in Section 4, only the condition $T_1(1) = 0$ is changed into $\psi_1(1) = 0$. The solution is completely analogical to that demonstrated in Section 4. Therefore let us write down only the final results. The quantities ρ_1, ρ_2 can be found from the equations

$$\rho_2 = \gamma\rho_1, \qquad Q\gamma(2-\rho_1^3-\rho_2^3) = 1+\gamma. \tag{5.1}$$

The dimensionless thicknesses of the carrying layers are

$$\begin{aligned} u^- &= 1+\tfrac{3}{2}Q(\rho_2^2-\rho^2), \quad \text{for } \rho\in[0_1\rho_2] \quad \text{and} \quad u^- \equiv 1, \quad \text{for } \rho\in[\rho_2, 1] \\ u^+ &= 1+\tfrac{3}{2}Q\gamma(\rho_1^2-\rho^2), \quad \text{for } \rho\in[0_1\rho_1] \quad \text{and} \quad u^+ \equiv 1, \quad \text{for } \rho\in[\rho_1, 1]. \end{aligned} \tag{5.2}$$

While increasing Q, quantity ρ_2 also increases. If

$$Q = Q_* = \gamma^2 \frac{\gamma+1}{\gamma^3-1},$$

then $\rho_2 = 1$, and the solution given above is not applicable for $Q > Q_*$. In this case régimes $D-A$ for $\rho \in [0_1 \rho_1)$ and $DE-A$ for $\rho \in (\rho_1, 1]$ appear. It can be shown that now quantity ρ_1 does not depend upon Q and has a constant value $\rho_1 = 1/\gamma$. The dimensionless thicknesses are as follows

$$\begin{aligned} u^- &= \tfrac{1}{2}Q(5-2\rho_1^3-3\rho^2)-\frac{1}{\gamma}, \quad \text{for } \rho \in [0, 1] \\ u^+ &= 1+\tfrac{3}{2}Q\gamma(\rho_1^2-\rho^2), \quad \text{for } \rho \in [0_1 \rho_1] \quad \text{and} \quad u^+ \equiv 1, \quad \text{for } \rho \in [\rho_1, 1]. \end{aligned} \tag{5.3}$$

6. DISCUSSION

When $\gamma = 1$ (the material has equal yield stresses in tension and compression) it follows from (4.1) and (5.1) that $\rho_1 = \rho_2$, $Q(1-\rho_1^3) = 1$ and the solutions achieved in Sections 4–5 coincide. For this case $u^- = u^+$ (the carrying layers have equal thicknesses). In the extreme case when $h_0 \to 0$ our formulae coincide with the results published before.

The effectiveness of the achieved solutions can be estimated by the values of integral (1.3). In view of the equations (4.2), (5.2) and (5.3) we obtain

$$J_1 = 1+\tfrac{3}{8}Q(\rho_1^4+\gamma\rho_2^4), \tag{6.1}$$

$$J_2 = \begin{cases} 1+\tfrac{3}{8}Q(\rho_2^4+\gamma\rho_1^4), & \text{for } Q \le Q_* \\ \dfrac{\gamma-1}{2\gamma}+\dfrac{1}{8}Q\left(7-\dfrac{1}{\gamma^3}\right), & \text{for } Q \ge Q_*. \end{cases} \tag{6.2}$$

Here J_1 and J_2 are the values of integral (1.3) for the cases $T_1(1) = 0$ and $v(1) = 0$ respectively. The calculations, which were carried out for the given values of Q, showed that $J_2 < J_1$; e.g. we have $J_1 = 1{\cdot}95$, $J_2 = 1{\cdot}48$ for $Q = 2$, $\gamma = 1{\cdot}5$ and $J_1 = 3{\cdot}06$, $J_2 = 2{\cdot}07$ for $Q = 2$, $\gamma = 3$. It follows from these data that for the minimum weight design the case $v(1) = 0$ is considerably more advantageous than the case $T_1(1) = 0$.

Now let us set a somewhat different problem: to design a sandwich plate with constant thicknesses of the carrying layers $h^+ \ge h_0$ and $h^- \ge h_0$ so that load q would have the greatest value for a given value of $h^- + h^+$.

The solution of this problem is in the case $T_1(1) = 0$ as follows

$$\begin{aligned} &h^+ = \gamma h^-, \qquad Q = \frac{h^-}{h_0} = u^- \\ &J_3 = \tfrac{1}{2}Q(1+\gamma). \end{aligned} \tag{6.3}$$

For the case $v(1) = 0$ one gets

$$\begin{aligned} &h^+ = h_0, \qquad Q = \frac{1}{2}\left(u^- + \frac{1}{\gamma}\right), \\ &J_4 = Q+\frac{\gamma-1}{2\gamma}. \end{aligned} \tag{6.4}$$

Here J_3 and J_4 are the values of integral (1.3), calculated for $h^\pm = \text{const}$.

It is of interest to compare the projects with variable and constant thicknesses of the carrying layers. The economy in weight for variable thicknesses is characterized by the quantities $\eta_1 = 1 - J_1/J_3$, $\eta_2 = 1 - J_2/J_4$. The values of η_1 and η_2 vs nondimensional load Q are presented for $\gamma = 1$, $\gamma = 1{\cdot}5$ and $\gamma = 3$ in Fig. 4. It follows from this figure that the economy in weight can be extended to 25 per cent.

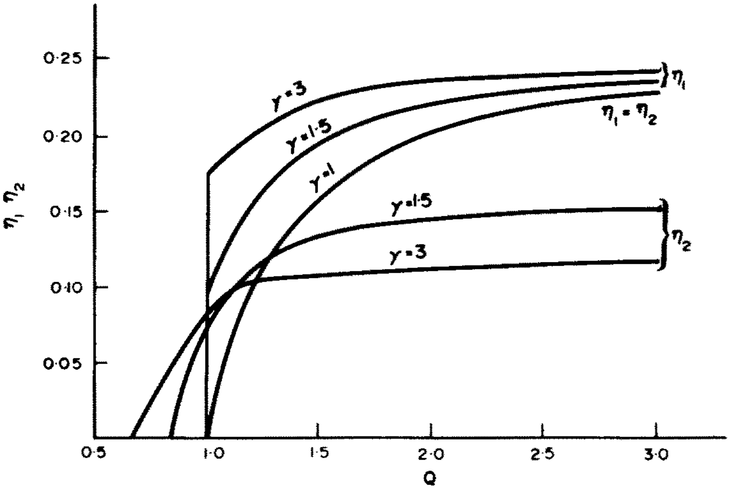

FIG. 4. Economy coefficients η_1, η_2 vs nondimensional load Q.

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Абстракт—Исследуется задача минимального веса круглых пластин типа "сэндвича" при дополнительном ограничении, что толщины несущих слоев не могут стать меньшими от заданной величины h_0. Материал пластины жестко-пластическмй, для условия текучести можно взять любое кусочно-линейное условие текучести.

Поставленная задача решается при помощи принципа максимума Л. С. Понтрягина. Найдутся всевозможные оптимальные режимы. Выясняется механический смысл сопряженной системы. Более подробно анализируется случай материала, имеющего различные пределы текучести при растяжении и сжатии. Составлены три проекта наименьшего веса для свободно опертой пластины, нагруженной равномерным поперечным давлением.